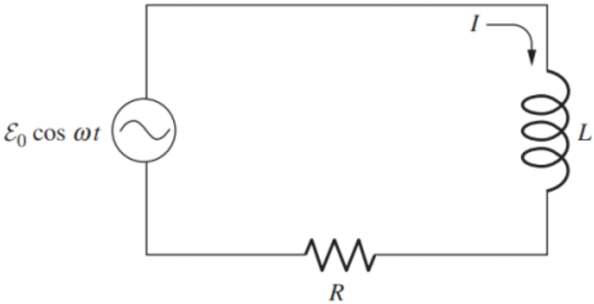
—Chapter 8—

Alternating-Current Circuits

8-1 Oscillator Circuit

A. RL CIRCUIT

(1) A circuit with inductance, driven by an alternating electromotive force.



The Kirchhoff loop equation for the series RL circuit and the alternating current is

$$L\frac{dI}{dt} + RI = \mathcal{E}_0 \cos \omega t$$

$$I(t) = I_0 \cos(\omega t + \phi)$$

Thus, we obtain

$$-\omega L I_0 \sin(\omega t + \phi) + R I_0 \cos(\omega t + \phi) = \mathcal{E}_0 \cos \omega t \cdots\cdots \text{(a)}$$

Using trigonometric angle sum identity, we obtain

$$-\omega L I_0(\sin \omega t \cos \phi + \cos \omega t \sin \phi) + R I_0(\cos \omega t \cos \phi - \sin \omega t \sin \phi) = \mathcal{E}_0 \cos \omega t$$

$$(-\omega L I_0 \cos \phi - R I_0 \sin \phi) \sin \omega t + (-\omega L I_0 \sin \phi + R I_0 \cos \phi - \mathcal{E}_0) \cos \omega t = 0$$

Setting the coefficients of $\sin \omega t$ and $\cos \omega t$ separately equal to zero gives, respectively,

$$-\omega L I_0 \cos \phi - R I_0 \sin \phi = 0 \Rightarrow \tan \phi = -\frac{\omega L}{R}$$

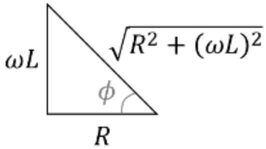
and

$$-\omega L I_0 \sin \phi + R I_0 \cos \phi - \mathcal{E}_0 = 0$$

which gives

$$
\begin{aligned}
I_0 &= \frac{\mathcal{E}_0}{R\cos\phi - \omega L \sin\phi} \\
&= \frac{\mathcal{E}_0}{R\cos\phi + R\tan\phi\sin\phi} \\
&= \frac{\mathcal{E}_0 \cos\phi}{R\left(\cos^2\phi + \sin^2\phi\right)} \\
&= \frac{\mathcal{E}_0 \cos\phi}{R}
\end{aligned}
$$

Since



$$
\cos\phi = \frac{R}{\sqrt{R^2 + (\omega L)^2}}
$$

we get

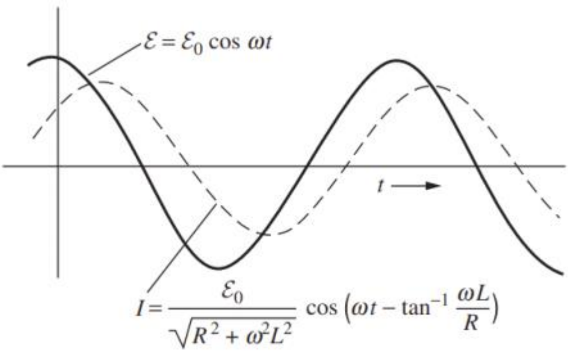
$$
I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L)^2}}
$$

Since ωL has the dimensions of resistance, this quantity is called the inductive resistance.

(2) Thus, we have

$$
\mathcal{E} = \mathcal{E}_0 \cos\omega t
$$

$$
I = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)
$$



Since ϕ is a negative angle, the current reaches its maximum a bit later than the electromotive force.

(3) Complex exponential solutions - A powerful and beautiful method

The Kirchhoff loop equation is

$$L\frac{dI}{dt} + RI = \mathcal{E}_0 e^{i\omega t}$$

$$I(t) = \tilde{I}_0 e^{i\omega t} \quad \text{where } \tilde{I}_0 \text{ is a complex number}$$

Thus, we obtain

$$i\omega L\tilde{I}_0 \cancel{e^{i\omega t}} + R\tilde{I}_0 \cancel{e^{i\omega t}} = \mathcal{E}_0 \cancel{e^{i\omega t}}$$

Canceling the $e^{i\omega t}$, we get

$$\begin{aligned}\tilde{I}_0 &= \frac{\mathcal{E}_0}{i\omega L + R} \\ &= \frac{\mathcal{E}_0}{R^2 + (\omega L)^2}(R - i\omega L) \\ &= \frac{\mathcal{E}_0}{R^2 + (\omega L)^2}\sqrt{R^2 + (\omega L)^2}e^{i\phi} \\ &= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L)^2}}e^{i\phi}\end{aligned}$$

$$\tan\phi = -\frac{\omega L}{R} \Rightarrow \phi = \tan^{-1}\left(-\frac{\omega L}{R}\right) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

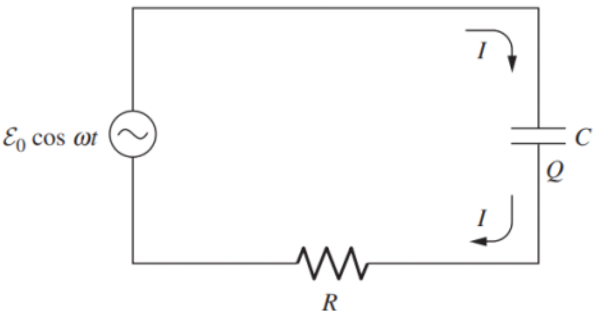
Thus, we obtain

$$\mathcal{E} = \mathcal{E}_0 e^{i\omega t}$$

$$I = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L)^2}} e^{i\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)}$$

B. RC CIRCUIT

(1) An alternating electromotive force in a circuit containing resistance and capacitance where we have defined Q to be the charge on the bottom plate of the capacitor.



The Kirchhoff loop equation for the series RC circuit and the alternating current is

$$-\frac{Q}{C} + RI = \mathcal{E}_0 e^{i\omega t}$$
$$Q = CV$$
$$I(t) = \tilde{I}_0 e^{i\omega t}$$

Since

$$I = -\frac{dQ}{dt}$$
$$Q = -\int I\,dt = -\frac{\tilde{I}_0}{i\omega} e^{i\omega t}$$

we have

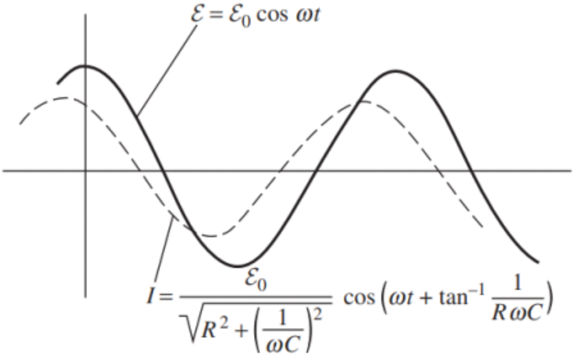
$$\frac{\tilde{I}_0}{i\omega C}\cancel{e^{i\omega t}} + R\tilde{I}_0\cancel{e^{i\omega t}} = \mathcal{E}_0\cancel{e^{i\omega t}}$$

Canceling the $e^{i\omega t}$, we get

$$\tilde{I}_0 = \frac{\mathcal{E}_0}{\frac{1}{i\omega C} + R} = \frac{\mathcal{E}_0}{R^2 + (1/\omega C)^2}\left(R + i\frac{1}{\omega C}\right) = \frac{\mathcal{E}_0}{\sqrt{R^2 + (1/\omega C)^2}} e^{i\phi}$$
$$\tan\phi = \frac{1/\omega C}{R} = \frac{1}{R\omega C}$$

(2) Thus, we obtain

$$\mathcal{E} = \mathcal{E}_0 e^{i\omega t}$$
$$I = \frac{\mathcal{E}_0}{\sqrt{R^2 + (1/\omega C)^2}} e^{i\left(\omega t + \tan^{-1}\frac{1}{R\omega C}\right)}$$

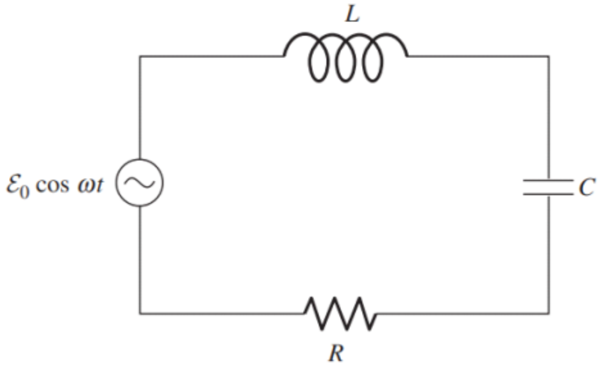


The maximum in I occurs here a little earlier than the maximum in $\mathcal{E}$.

C. RLC CIRCUIT

(1) The RLC circuit driven by a sinusoidal electromotive force where we

have defined Q to be the charge on the bottom plate of the capacitor.



The Kirchhoff loop equation for the series RLC circuit is

$$L\frac{dI}{dt} - \frac{Q}{C} + RI = \mathcal{E}_0 e^{i\omega t}$$

$I(t) = \tilde{I}_0 e^{i\omega t}$ where $\tilde{I}_0$ is a complex number

$$Q = -\int I\, dt = -\frac{\tilde{I}_0}{i\omega} e^{i\omega t}$$

Thus, we obtain

$$i\omega L\tilde{I}_0 \cancel{e^{i\omega t}} + \frac{\tilde{I}_0}{i\omega C}\cancel{e^{i\omega t}} + R\tilde{I}_0 \cancel{e^{i\omega t}} = \mathcal{E}_0 \cancel{e^{i\omega t}}$$

Canceling the $e^{i\omega t}$, we get

$$i\omega L\tilde{I}_0 + \frac{\tilde{I}_0}{i\omega C} + R\tilde{I}_0 = \mathcal{E}_0$$

$$\tilde{I}_0 = \frac{\mathcal{E}_0}{i\omega L + R + \frac{1}{i\omega C}}$$

$$= \frac{\mathcal{E}_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left(R - i\left(\omega L - \frac{1}{\omega C}\right)\right)$$

$$= \frac{\mathcal{E}_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{i\phi}$$

$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\phi}$$

$$\tan\phi = \frac{\frac{1}{\omega C} - \omega L}{R} = \frac{1}{R\omega C} - \frac{\omega L}{R}$$

(2) Thus, we obtain

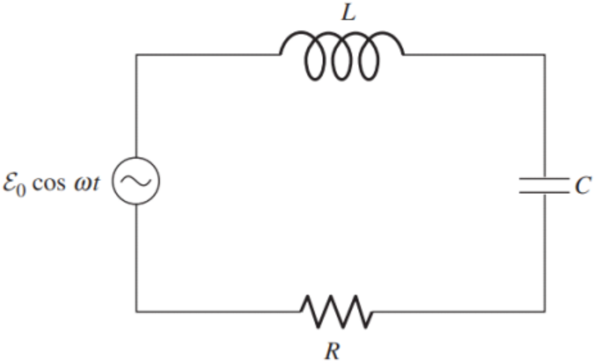
$$\mathcal{E} = \mathcal{E}_0 e^{i\omega t}$$

$$I = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\left(\omega t + \tan^{-1}\left(\frac{1}{R\omega C} - \frac{\omega L}{R}\right)\right)}$$

8-2 Resonant Circuit

A. RLC CIRCUIT

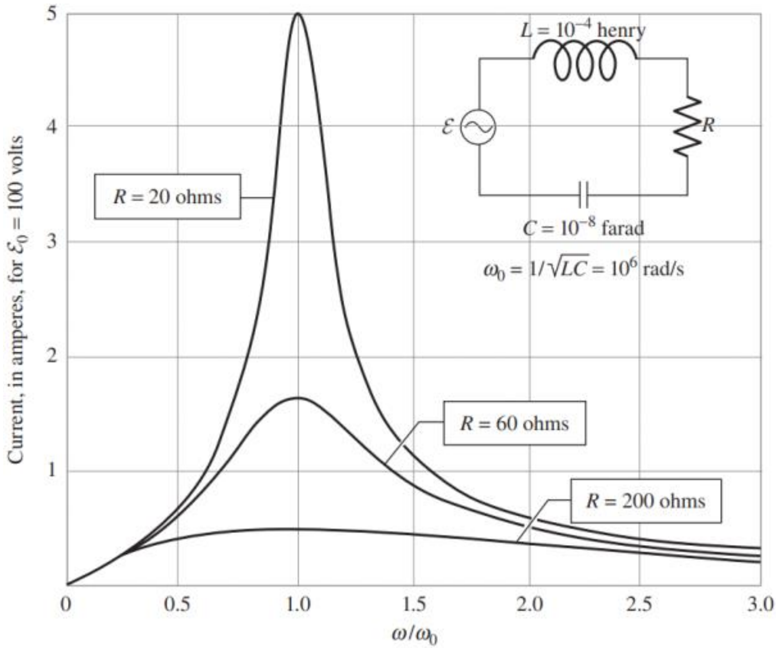
(1) The RLC circuit driven by a sinusoidal electromotive force.



We have

$$\mathcal{E} = \mathcal{E}_0 e^{i\omega t}$$

$$I = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\left(\omega t + \tan^{-1}\left(\frac{1}{R\omega C} - \frac{\omega L}{R}\right)\right)}$$



(2) The maximum in $I_0(\omega)$ occurs at

$$\omega L - \frac{1}{\omega C} = 0 \Rightarrow \omega^2 = \frac{1}{LC} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

Thus, we obtain

$$\mathcal{E}_{\max} = \mathcal{E}_0$$

$$I_{0,\max} = \frac{\mathcal{E}_0}{\sqrt{R^2 + 0^2}} = \frac{\mathcal{E}_0}{R}$$

(3) Width of the $I_0(\omega)$ curve

The width of a resonance peak is the full width, $2\Delta\omega$, between half-power points, i.e.,

$$P_{1/2} = \frac{1}{2} P_{\max}$$

Since

$$P = I^2 R$$

$$\Rightarrow I_{0,1/2} = \frac{1}{\sqrt{2}} I_{0,\max} = \frac{\mathcal{E}_0}{\sqrt{R^2 + R^2}}$$

$$\Rightarrow \left| \omega L - \frac{1}{\omega C} \right| = R$$

Consider frequencies in the neighborhood of ω_0,

$$\omega = \omega_0 + \Delta\omega = \omega_0\left(1 + \frac{\Delta\omega}{\omega_0}\right)$$

To first order in $\Delta\omega/\omega_0$,

$$\omega L - \frac{1}{\omega C} = \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0 C\left(1 + \frac{\Delta\omega}{\omega_0}\right)}$$

$$\approx \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0 C}\left(1 - \frac{\Delta\omega}{\omega_0}\right)$$

Since

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 L = \frac{1}{\omega_0 C}$$

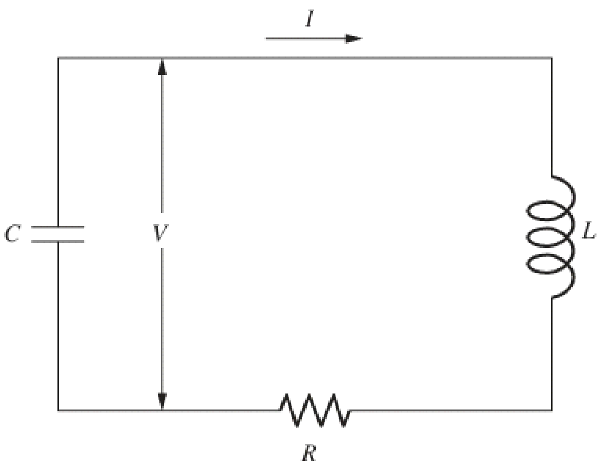
thus, we obtain

$$\omega L - \frac{1}{\omega C} = \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \omega_0 L\left(1 - \frac{\Delta\omega}{\omega_0}\right) = \omega_0 L\frac{2\Delta\omega}{\omega_0} = R$$

$$\Rightarrow \frac{2|\Delta\omega|}{\omega_0} = \frac{R}{\omega_0 L} = R\omega_0 C$$

B. DAMPED OSCILLATOR CIRCUIT

(1) A series RLC circuit.



$$L\frac{dI}{dt} + RI - \frac{Q}{C} = 0$$

$$Q = CV$$

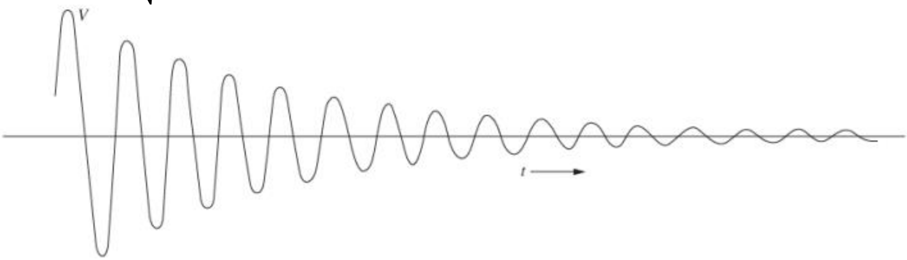
$$I = -\frac{dQ}{dt} = -C\frac{dV}{dt}$$

Thus, we have

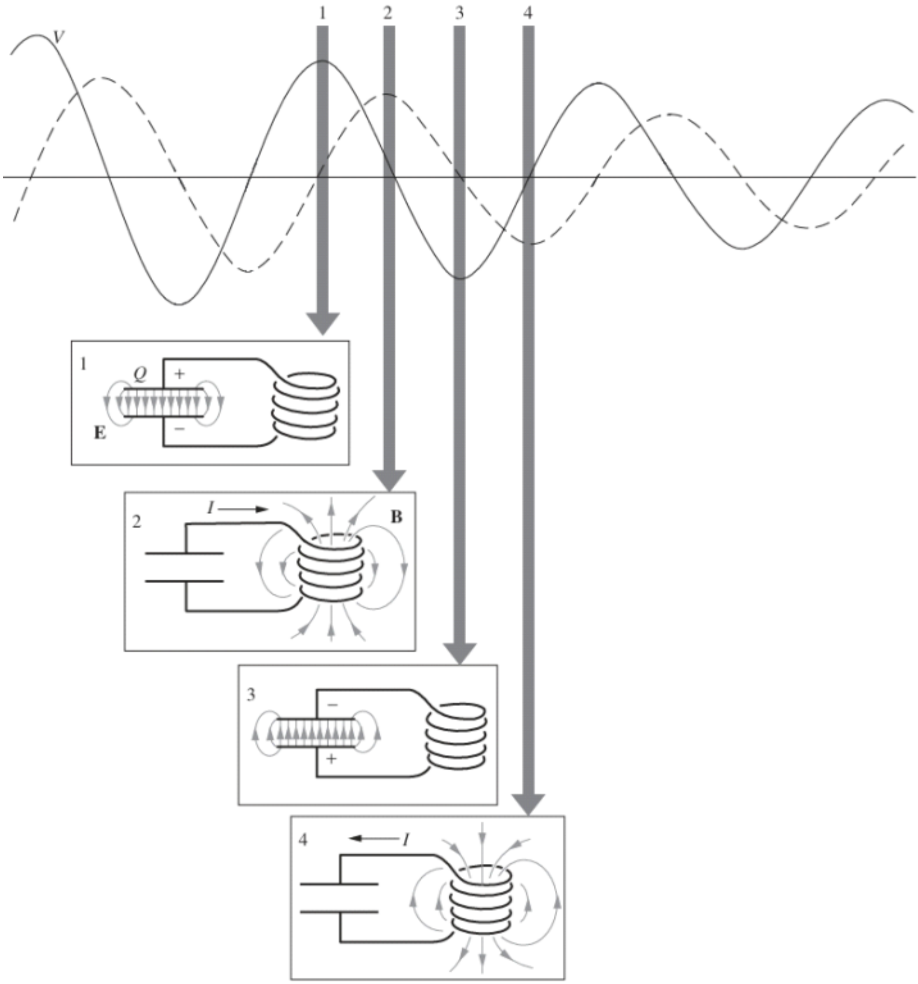
$$L\frac{d}{dt}\left(-C\frac{dV}{dt}\right)+R\left(-C\frac{dV}{dt}\right)-V=0$$
$$\frac{d^2V}{dt^2}+\frac{R}{L}\frac{dV}{dt}+\frac{1}{LC}V=0$$
This is a damped oscillation equation. We shall try a solution of the form
$$V(t)=Ae^{-\alpha t}e^{i\omega t}$$
Since
$$\frac{dV}{dt}=Ae^{-\alpha t}e^{i\omega t}(-\alpha+i\omega)$$
$$\frac{d^2V}{dt^2}=Ae^{-\alpha t}e^{i\omega t}\left(\alpha^2-2i\alpha\omega-\omega^2\right)$$
thus, we obtain
$$Ae^{-\alpha t}e^{i\omega t}\left(\alpha^2-2i\alpha\omega-\omega^2\right)+\frac{R}{L}Ae^{-\alpha t}e^{i\omega t}(-\alpha+i\omega)+\frac{1}{LC}Ae^{-\alpha t}e^{i\omega t}=0$$
$$\left(\alpha^2-2i\alpha\omega-\omega^2\right)+\frac{R}{L}(-\alpha+i\omega)+\frac{1}{LC}=0$$
$$\alpha^2-\omega^2-\alpha\frac{R}{L}+\frac{1}{LC}-i\left(2\alpha\omega-\frac{R\omega}{L}\right)=0$$
We require
$$2\alpha\omega-\frac{R\omega}{L}=0\Rightarrow\alpha=\frac{R}{2L}$$
$$\alpha^2-\omega^2-\alpha\frac{R}{L}+\frac{1}{LC}=0\Rightarrow\omega^2=\frac{1}{LC}-\frac{R^2}{4L^2}$$

(2) Thus, we obtain
$$V(t)=Ae^{-\frac{R}{2L}t}e^{i\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}t}$$

$$
\begin{aligned}
I &= -C\frac{dV}{dt} \\
&= ACe^{-\alpha t}e^{i\omega t}(\alpha - i\omega) \\
&= ACe^{-\alpha t}\sqrt{\alpha^2+\omega^2}\,e^{i(\omega t - \tan^{-1}\omega/\alpha)} \\
&= ACe^{-\frac{R}{2L}t}\sqrt{\frac{R^2}{4L^2}+\frac{1}{LC}-\frac{R^2}{4L^2}}\,e^{i\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}t-\tan^{-1}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\Big/\frac{R}{2L}\right)} \\
&= A\sqrt{\frac{C}{L}}e^{-\frac{R}{2L}t}e^{i\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}t-\tan^{-1}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\Big/\frac{R}{2L}\right)}
\end{aligned}
$$



Zoom in a period:



(3) If $R = 0$,

$$V(t) = Ae^{-\frac{0}{2L}t}e^{i\sqrt{\frac{1}{LC}-\frac{0}{4L^2}}t} = Ae^{i\omega_0 t} \text{ where } \omega_0 = \frac{1}{\sqrt{LC}}$$

$$I(t) = A\sqrt{\frac{C}{L}}e^{-\frac{0}{2L}t}e^{i\left(\sqrt{\frac{1}{LC}-\frac{0}{4L^2}}t-\tan^{-1}\sqrt{\frac{1}{LC}-\frac{0}{4L^2}}\Big/\frac{0}{2L}\right)}$$

$$= A\sqrt{\frac{C}{L}}e^{i\left(\sqrt{\frac{1}{LC}}t-\tan^{-1}\infty\right)}$$

$$= A\sqrt{\frac{C}{L}}e^{i(\omega_0 t-\pi/2)}$$

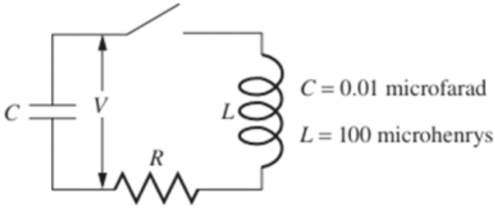
we obtain a undamped oscillator.

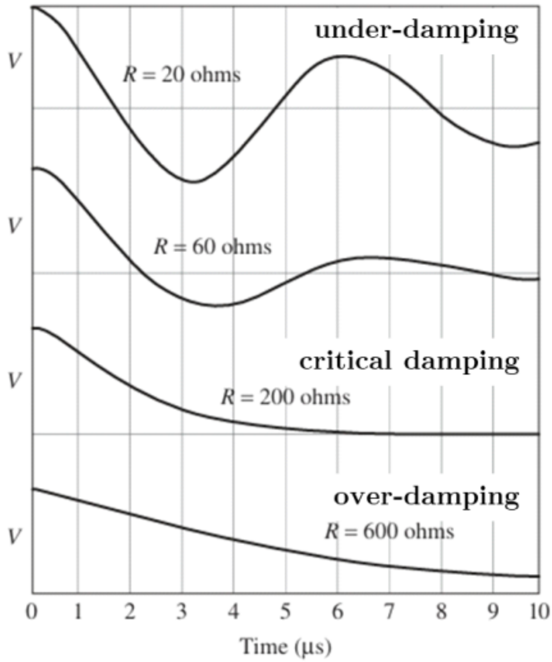
(4) If $\frac{1}{LC}-\frac{R^2}{4L^2}=0 \Rightarrow R = 2\sqrt{\frac{L}{C}}$,

$$V(t) = Ae^{-\frac{R}{2L}t}e^{i0t} = Ae^{-\frac{1}{\sqrt{LC}}t} = Ae^{-\omega_0 t}$$

$$I(t) = A\sqrt{\frac{C}{L}}e^{-\frac{R}{2L}t}e^{i\left(0t-\tan^{-1}0/\frac{R}{2L}\right)} = A\sqrt{\frac{C}{L}}e^{-\frac{R}{2L}t} = A\sqrt{\frac{C}{L}}e^{-\omega_0 t}$$

we obtain the critical damping.

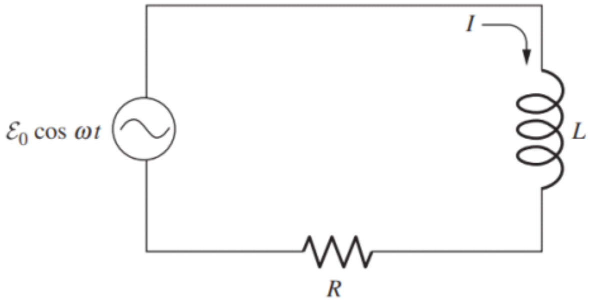

C
V
L
R
C = 0.01 microfarad
L = 100 microhenrys


under-damping
R = 20 ohms
R = 60 ohms
critical damping
R = 200 ohms
over-damping
R = 600 ohms
V
V
V
V
0
1
2
3
4
5
6
7
8
9
10
Time (μs)

8-3 Power and Energy in Alternating-Current Circuits

A. ADMITTANCE AND IMPEDENCE

(1) The relation between current flow in a circuit element and the voltage across the element can be expressed as a relation between the complex numbers that represent the voltage and the current.



The voltage and current oscillation are represented by

$$\tilde{V} = \mathcal{E}_0 e^{i\omega t}$$

$$\tilde{I} = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L)^2}} e^{i(\omega t + \phi)} = \frac{e^{i\phi}}{\sqrt{R^2 + (\omega L)^2}} \mathcal{E}_0 e^{i\omega t}$$

$$\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

(2) We define a complex number as follows,

$$\tilde{I} = Y\tilde{V}$$

$$Y = \frac{e^{i\phi}}{\sqrt{R^2 + (\omega L)^2}} \cdots\cdots \text{admittance (conductivity in complex)}$$

$$= \frac{1}{i\omega L + R}$$

The same relation can be expressed with the reciprocal of Y, denoted by Z,

$$\tilde{V} = Z\tilde{I}$$

$$Z = \frac{1}{Y} \cdots\cdots \text{impedence (resistivity in complex)}$$

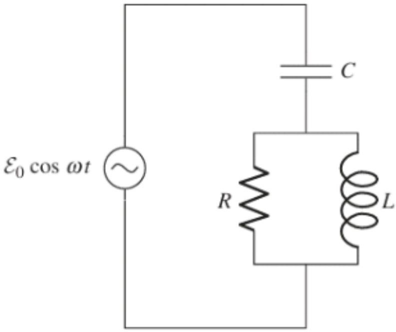
(3) The properties of the three basic circuit elements are summarized

Complex impedances

Symbol	Admittance, Y	Impedance, $Z = 1/Y$
R	$\frac{1}{R}$	R
L	$\frac{1}{i\omega L}$	$i\omega L$
C	$i\omega C$	$\frac{1}{i\omega C}$
	$I = YV$	$V = ZI$

EXAMPLES:

1. Find the complex voltage across, and current through, each of the three elements where $L = R/\omega$ and $C = 1/\omega R$.



ANSWER:

The three impedances are then

$$Z_C = \frac{1}{i\omega C} = -iR$$
$$Z_R = R$$
$$Z_L = i\omega L = iR$$

The impedance of the entire circuit is

$$Z = Z_C + \frac{1}{\frac{1}{Z_R} + \frac{1}{Z_L}} = \frac{1}{i\omega C} + \frac{1}{\frac{1}{R} + \frac{1}{i\omega L}} = -iR + \frac{1}{\frac{1}{R} + \frac{1}{iR}} = R\frac{1-i}{2}$$

The complex total current:

$$\tilde{V}_{\mathcal{E}} = Z\tilde{I} \Rightarrow \tilde{I} = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{R}\frac{1}{1-i} = \frac{\mathcal{E}_0}{R}(1+i)$$

The complex voltage across each element:

$$\tilde{V}_C = Z_C\tilde{I} = (-iR)\frac{\mathcal{E}_0}{R}(1+i) = \mathcal{E}_0(1-i)$$
$$\tilde{V}_R = \tilde{V}_L = \mathcal{E}_0 - \tilde{V}_C = \mathcal{E}_0 - \mathcal{E}_0(1-i) = i\mathcal{E}_0$$

The complex current through each element:

$$\tilde{I}_C = \tilde{I}$$

$$\tilde{I}_R = \frac{\tilde{V}_R}{Z_R} = \frac{i\mathcal{E}_0}{R}$$

$$\tilde{I}_L = \frac{\tilde{V}_L}{Z_L} = \frac{i\mathcal{E}_0}{iR} = \frac{\mathcal{E}_0}{R}$$

B. POWER AND ENERGY IN ALTERNATING-CURRENT CIRCUITS

(1) The energy dissipated in the resistor is given by

$$P_R = RI^2 = \frac{V^2}{R}$$

However, as $I = \tilde{I}_0 e^{i\omega t}$ and $V = V_0 e^{i\omega t}$, we shall evaluate the average power instead of the instantaneous power.

Since

$$\overline{V^2} = \overline{(VV)} = \Re\frac{1}{2}VV^* = \Re\frac{1}{2}V_0 e^{i\omega t}V_0 e^{-i\omega t} = \frac{V_0^2}{2}$$

or take the real part only

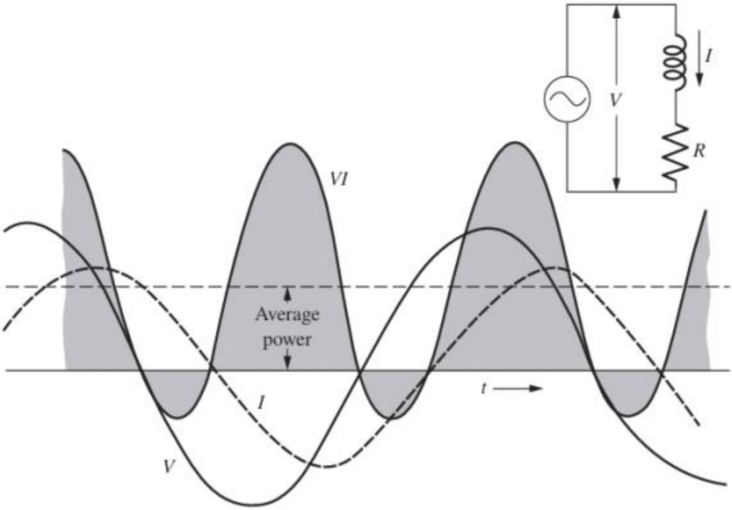
$$\frac{V_0^2}{T}\int_0^T \cos^2\omega t\,dt = \frac{V_0^2}{T}\int_0^T\left(\frac{1}{2}+\frac{1}{2}\cos 2\omega t\right)dt = \frac{V_0^2}{2T}T = \frac{V_0^2}{2}$$

The average power dissipated in the resistor is

$$\bar{P}_R = \frac{1}{2}\frac{V_0^2}{R} = \frac{V_{\mathrm{rms}}^2}{R}$$

where

$$V_{\mathrm{rms}} = \frac{V_0}{\sqrt{2}} \cdots\cdots \text{ root-mean-square (rms) value}$$



(2) The average power $\bar{P}$ delivered to the RL circuit corresponds to the horizontal dashed line.

$$VI = V_0 e^{i\omega t} \tilde{I}_0 e^{i\omega t} = V_0 \tilde{I}_0 e^{2i\omega t}$$

$$\bar{P} = \overline{\left(V\tilde{I}\right)} = \Re \frac{1}{2} V_0 \tilde{I}_0^* e^{i\omega t} e^{-i\omega t} = \frac{1}{2} V_0 I_0 \cos\phi = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos\phi$$